Continuous fields of C*-algebras and their invariants

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$C(X)$-algebras and Continuous fields

X metrizable compact space

Definition. A structure of $C(X)$-algebra on a separable C*-algebra A consists of a unital $*$-homomorphism

$$\theta : C(X) \to Z(\mathbb{M}(A))$$

θ is called the structure morphism. If A is unital $\theta : C(X) \to Z(A)$, $Z(A)$=the center of A. Usually we drop θ from notation and write

$$\theta(f)a = fa$$

Putting a $C(X)$-algebra structure on A is equivalent to giving a continuous map $\Theta : \mathrm{Prim}(A) \to X$.

Due to the $C(X)$-module structure we can localize at the points of X:

fibers: $A(x) = A/C(X,x)A$ restrictions $A(Y) = A/C(X,Y)A$

evaluation maps: $\pi_x : A \to A(x)$, $a(x) := \pi_x(a)$.

A $C(X)$-algebra becomes a continuous field **if**

$x \mapsto \|a(x)\|$ is continuous $\forall a \in A$.

Putting a continuous field structure on A is equivalent to giving a continuous and open map $\Theta : \mathrm{Prim}(A) \to X$.

If a $C(X)$-algebra A is unital and all fibers are simple then A is automatically a continuous field.

Morphisms of continuous fields

A morphism of continuous fields is a $C(X)$-linear $*$-homomorphism $\varphi : A \to B$, $\varphi(fa) = f\varphi(a)$.

It induces maps $\varphi_x : A(x) \to B(x)$ on each fiber

$\varphi_V : A(V) \to A(V)$ where $V \subset X$ is either open or closed.

V open $\Rightarrow A(V) := C_0(V)A$ ideal of A

V closed $\Rightarrow A(V) := C(X,V)A$ quotient of A

Examples of continuous fields

If A is any separable C*-algebra with center $Z(A) = C(X)$ and each $A(x)$ fiber is simple then A is always a continuous field over X for the structure map $Z(A) \subset A$.

(0) If the primitive spectrum $Prim(A) = X$ of A is Hausdorff, then A is a continuous field with simple fibers over X.

(1) Trivial field with fiber D: $A = C(X) \otimes D$.

(2) Locally trivial fields: $\forall x \in X$ has closed neighborhood V with $A(V) \cong C(V) \otimes D$.

(3) next slide will show fields with fixed fiber but which are not locally trivial

How complicated can a continuous field be?

If $\gamma : D \to D$ injective $*$-hom and $0 \leq r \leq 1$, then

$$B_r = \{f \in C[0,1] \otimes D : f(r) \in \gamma(D)\}$$

has all its fibers are $\cong D$. The fiber at r is $\gamma(D) \cong D$.

Fact: $B_r \cong C[0,1] \otimes D$ if and only if there is a continuous path $\alpha : [0,1] \to \mathrm{End}(D)$ such that $\alpha[0,1) \subset \mathrm{Aut}(D)$ and $\alpha(1) = \gamma$.

A field nowhere locally trivial

$$B_{r_i} = \{f \in C[0,1] \otimes D : f(r_i) \in \gamma(D)\}$$

If (r_n) dense in $[0,1]$, then the infinite tensor product

$$B_{r_1} \otimes_{C[0,1]} B_{r_2} \otimes_{C[0,1]} \cdots B_{r_3} \otimes_{C[0,1]} \cdots$$

is field over $[0,1]$ which does not need to be locally trivial at any point,

even though all fibers are mutually isomorphic to $D \otimes D \otimes D \otimes \cdots$.

being a continuous field $\not\Rightarrow$ any kind of local triviality

even if the fibers are mutually isomorphic.

Tame fibers

Thm. (Fell, Tomiyama-Takesaki) a unital C*-algebra all of whose irreducible representations are of the same finite dimension n is given by the continuous sections of a locally trivial bundle over a compact Hausdorff space with fiber $M_n(\mathbb{C})$.

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What if all primitive quotients of A are isomorphic to $\mathcal{K}$ the compact operators on a separable Hilbert space and $\mathrm{Prim(A)} = X$ is Hausdorff? A is certainly a continuous field over X with fiber $\mathcal{K}$. But is it locally trivial?

What if for all irred reps π of A, $\pi(A) \cong D$, same D.

Review of Dixmier-Douady theory

A=separable continuous field with fibers the compact operators $\mathcal{K}$ over a compact space X.

Def A satisfies Fell's condition if $\forall\, x \in X\ \exists\, p \in A$ and a neighborhood V of x such that $p(v)$ is a projection of rank one for each point $\forall\, v \in V$.

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How to extend these results to fields with all fibers isom. to a fixed D?

A continuous field which does not satisfy Fell's condition

$$B = \{f \in C[0,1] \otimes M_2(\mathbb{C}) : f(1/2) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\}$$

$B \otimes \mathcal{K}$ has all its fibers isomorphic to $\mathcal{K}$ the compact operators.

$B \otimes \mathcal{K} \ncong C[0,1] \otimes \mathcal{K}$. Indeed, since $\mathrm{Ker}\,(B \to B(1/2))$ is contractible $\Rightarrow K_0(B) \cong K_0(B(1/2))$.

Thus $(\pi_x)_* : \mathbb{Z} = K_0(B) \to K_0(B(x)) = \mathbb{Z}$ is $k \mapsto 2k$ if $x \neq 1/2$.

It useful to consider the K-theory presheaf to capture invariants. $K_0(A[0,2/3]) = \mathbb{Z} \to K_0(A[0,1/3]) = \mathbb{Z}$ is again $k \mapsto 2k$.

Glueing and sheaves

Any $C(X)$-algebra yields a sheaf of C*-algebras

$V \mapsto A(V)$, $V \subset W \Rightarrow A(W) \to A(V)$. Moreover the following diagram is a pullback:

$$\begin{array}{ccc} A(U \cup V) & \longrightarrow & A(U) \\ \downarrow & & \downarrow \\ A(V) & \longrightarrow & A(U \cap V) \end{array}$$

$A(U \cup V) \cong \{(a, b) \in A(U) \oplus A(V) | \pi_{U \cap V}(a) = \pi_{U \cap V}(b)\}$.

Here U, V, W are closed subsets of X.

K-theory presheaf

$V \mapsto K_0(B(V))$, $\quad V \subset W \Rightarrow K_0(B(W)) \to K_0(B(V))$.

$$\begin{array}{ccc} K_*(A(U \cup V)) & \longrightarrow & K_*(A(U)) \\ \downarrow & & \downarrow \\ K_*(A(V)) & \longrightarrow & K_*(A(U \cap V)) \end{array}$$

is not a pullback in general (Mayer-Vietoris). It is a pullback in the case of sheaves (by definition).

One parameter continuous fields

Thm. (D-Elliott) All unital separable continuous fields over $[0,1]$ with fiber $\mathcal{O}_n$ are trivial.

Thm. (D-Elliott): The unital separable continuous fields over $[0,1]$ whose fibers are stable Kirchberg algebras satisfying the UCT with K_0 torsion free and $K_1 = 0$ are classified by the K-theory sheaf.

Thm. (D-Niu-Elliott): All continuous sheaves of countable abelian groups over $[0,1]$ arise as the K_0-sheaf of a one-parameter continuous field of Kirchberg algebras.

$\mathcal{O}_n$-fields

Thm: A unital separable continuous field over X finite dimensional compact Hausdorff and $A(x) \cong \mathcal{O}_n\ \forall x \in X$. Then A is locally trivial.

$n = 2 \quad \Rightarrow \quad A \cong C(X) \otimes \mathcal{O}_2$

$n = \infty \quad \Rightarrow \quad A \cong C(X) \otimes \mathcal{O}_\infty$

$3 \leq n < \infty \quad \Rightarrow \quad A \cong C(X) \otimes \mathcal{O}_n$ iff $(n-1)[1_A] = 0$ in $K_0(A)$.

Key property of O_n which explains the result: $\mathrm{Aut}(O_n)$ is a "deformation retract" of $\mathrm{End}_1(O_n)$. This is reminiscent of $\mathrm{End}_1(M_n(\mathbb{C})) = \mathrm{Aut}(M_n(\mathbb{C}))$

Fiberwise KK-theory

Kasparov defined $KK_X(A, B)$ for $C(X)$-algebras,

same definition as for KK

(E, T) is Fredholm-bimodule:

$[T, a] \sim 0$, $(T^2 - 1)a \sim 0$, $(T^* - T)a \sim 0$ and

$\boxed{f a \xi b = a \xi f b}$ $\quad a \in A$, $b \in B$, $f \in C(X)$.

product $KK_X(A, B) \times KK_X(B, C) \to KK_X(A, C)$

invertible elements $KK_X(A, B)^{-1}$

Have map $KK_X(A, B) \longrightarrow KK(A, B)$.

$$KK_X(A,B) \longrightarrow KK(A(V),B(V))$$

$\forall V \subset X$ closed or open.

Kirchberg's isomorphism theorem

Let A,B unital sep. nuclear C*-algebras with Hausdorff spectrum X.

Thm. (Kirchberg) $A \otimes \mathcal{O}_\infty \otimes \mathcal{K} \cong B \otimes \mathcal{O}_\infty \otimes \mathcal{K} \Leftrightarrow KK_X(A,B)^{-1} \neq \emptyset$.

Caveats:

$KK_X(A,B)$ is a complex object; very hard to compute. Even if $X = [0,1]$.

Given A and B how does one determine if they are KK_X-equivalent, i.e. if $KK_X(A,B)^{-1} \neq \emptyset$?

KK_X-equivalence

Thm. Let A and B be separable nuclear continuous fields over a finite dimensional compact metrizable space X. If $\sigma \in KK_X(A, B)$, then $\sigma \in KK_X(A, B)^{-1}$ if and only if $\sigma_x \in KK(A(x), B(x))^{-1}$ for all $x \in X$.

Condition is easy to verify if UCT is assumed for fibers.

Indeed, this amounts to the bijectivity of

$(\sigma_x)_* : K_*(A(x)) \to K_*(B(x))$.

Suppose now that $A = C(X) \otimes D$. Then

$$KK_X(C(X) \otimes D, B) \cong KK(D, B)$$

$$\sigma \in KK_X(C(X) \otimes D, B)^{-1} \Leftrightarrow \sigma_x \cong KK(D, B(x))^{-1}, \forall x \in X$$

Local triviality

Def. A satisfies the Fell condition if $\forall\, x \in X\ \exists\, V$ closed neighborhood of x and $\sigma \in KK(D, A(V))$ such that $\sigma_x \in KK(D, A(v))^{-1}$ for all $v \in V$.

Thm. X finite dimensional space. A separable unital continuous field with fibers stable Kirchberg algebras. Then A is locally trivial $\Leftrightarrow$ A satisfies the Fell condition

Def. A satisfies the global Fell condition if $\exists\, \sigma \in KK(D, A)$ such that $\sigma_x \in KK(D, A(x))^{-1}$ for all $x \in X$. Suppose X finite dimensional and D stable Kirchberg algebra.

Cor. A satisfies the global Fell condition $\Leftrightarrow$ A is isomorphic to $C(X) \otimes D$.

Note that if $D = \mathcal{K}$, then we just have essentially the classic Fell condition. Indeed,

$$K_X(C(X) \otimes \mathcal{K}, A) \cong KK(\mathcal{K}, A) = \mathrm{Hom}(\mathbb{Z}, K_0(A))$$

Thus $\sigma \in K_X(C(X) \otimes \mathcal{K}, A)^{-1}$ is given by a virtual projection of virtual rank one in each fiber $K_0(A(x))$.

Conclusion: Thm 1 of Dixmier-Douady extends to general nuclear continuous fields.

What about Thm2 of Dixmier-Douady? How to classify locally trivial continuous fields with fiber D?

Homotopy groups

$\mathrm{Aut}(\mathcal{K}) = U(H)/\mathbb{T}$. Since $U(H)$ is contractible, all homotopy groups of $\mathrm{Aut}(\mathcal{K})$ are zero except for $\pi_2\mathrm{Aut}(\mathcal{K}) = \mathbb{Z}$.

Thus all homotopy groups of $B\mathrm{Aut}(\mathcal{K})$ are zero except for $\pi_3 B\mathrm{Aut}(\mathcal{K}) = \mathbb{Z}$. It follows that

$$[X, B\mathrm{Aut}(\mathcal{K})] \cong H^3(X, \mathbb{Z}).$$

The question is now to compute the homotopy classes $[X, B\mathrm{Aut}(D)]$ for a (unital) Kirchberg algebra.

Homotopy groups

If D is a unital C*-algebra,we let $C_\nu D$ denote the mapping cone C*-algebra of the unital map $\nu : \mathbb{C} \to D$.

Thm. Let D be a unital Kirchberg algebra. Suppose that X is path connected and finite dimensional and let $x_0 \in X$. Then there are bijections

$$\chi : [\, X, \mathrm{Aut}(D)^0] \to KK(C_\nu D, SC(X, x_0) \otimes A),$$

Cor $\pi_{2k+1}(\mathcal{O}_{m+1}) = \mathbb{Z}/m$, $\pi_{2k}(\mathcal{O}_{m+1}) = 0, k \geq 0$.

$0 \quad \mathbb{Z}/m \quad 0 \quad \mathbb{Z}/m \quad 0 \quad \mathbb{Z}/m \quad 0 \cdots$

$\mathrm{Aut}(O_{m+1})$

The computation of $[X, B\mathrm{Aut}(O_{m+1})]$ is still open. We know

$$[X, \mathrm{Aut}(O_{m+1})] \cong K_1(C(X) \otimes O_{m+1}).$$

Specifically the unitary $x \mapsto U(x) \in U(O_{m+1})$ gives a map

$X \mapsto \mathrm{End}(O_{m+1})$, $x \mapsto \alpha_x$,

$\alpha_x(S_i) = u(x)S_i$.

Then the map $x \mapsto \alpha_x$ is homotopic to a map $X \to \mathrm{Aut}(O_{m+1})$.

This yields the classification of unital O_{m+1} continuous fields over SX the suspension of X.

∞-dimensional spaces

Thm. Let A, B be nuclear, separable continuous fields over a finite dimensional metrizable spectrum X. Then an element $\sigma \in KK_X(A, B)$ is a KK_X-equivalence $\Leftrightarrow$ $\sigma_x \in KK(A(x), B(x))^{-1}$ for all $x \in X$.

Cor. Automatic triviality of separable continuous fields with fiber $\mathcal{O}_2 \otimes \mathcal{K}$ over finite dimensional spaces.

Example: There is a nontrivial separable unital continuous field A over X=the Hilbert cube with fibers isomorphic to $\mathcal{O}_2$; $A \not\cong C(X) \otimes \mathcal{O}_2$ despite the contractibility of X and $\pi_n \mathrm{Aut}(\mathcal{O}_2) = 0$ $\forall n \geq 0$.

Distinguishing invariant:

$K_0(A) = \oplus_{k=1}^{\infty} \mathbb{Z}/2$ whereas $K_0(C(X) \otimes \mathcal{O}_2) = 0$

∞-dimensional spaces

Using E_X-theory over non Hausdorff spaces:

Thm.(D-Meyer) Let A, B be nuclear, separable continuous fields over a metrizable compact space X. Then an element $\sigma \in KK_X(A,B)$ is a KK_X-equivalence $\Leftrightarrow$ $\sigma_x \in KK(A(V),B(V))^{-1}$ for all $V \subset X$ closed subset.

Cor.(D-Meyer) Let A be separable continuous field of simple nuclear algebras over a compact space X. Suppose that all ideals of A are KK-equivalent to 0. Then $A \otimes \mathcal{O}_\infty \otimes \mathcal{K} \cong C(X) \otimes \mathcal{O}_2 \otimes \mathcal{K}$.

UCT for zero dimensional Haudorff spaces

$$UCT: \quad \mathsf{Ext}_{\mathbb{Z}}\big(\mathsf{K}_*(A), \mathsf{K}_*(SB)\big) \rightarrowtail \mathsf{KK}(A,B) \twoheadrightarrow \mathsf{Hom}_{\mathbb{Z}}\big(\mathsf{K}_*(A), \mathsf{K}_*(B)\big).$$

$$UMCT: \quad \mathsf{PExt}_{\mathbb{Z}}\big(\mathsf{K}_*(A), \mathsf{K}_*(SB)\big) \rightarrowtail \mathsf{KK}(A,B) \twoheadrightarrow \mathsf{Hom}_{\Lambda}\big(\underline{\mathsf{K}}(A), \underline{\mathsf{K}}(B)\big).$$

Fact:

$$\mathsf{PExt}_{\mathbb{Z}}\big(\mathsf{K}_*(A), \mathsf{K}_*(B)\big) = \mathsf{Ext}_{\Lambda}\big(\underline{\mathsf{K}}(A), \underline{\mathsf{K}}(B)\big)$$

UCT for zero dimensional Haudorff spaces

Let X be a metrizable zero dimensional compact Hausdorff space

A, B nuclear continuous fields over X with fibers satisfying the UCT.

The following sequence is not exact.

$$\mathsf{Ext}_{\mathsf{C}(X,\mathbb{Z})}\big(\mathsf{K}_*(A), \mathsf{K}_*(SB)\big) \rightarrowtail \mathsf{KK}_X(A,B) \twoheadrightarrow \mathsf{Hom}_{\mathsf{C}(X,\mathbb{Z})}\big(\mathsf{K}_*(A), \mathsf{K}_*(B)\big).$$

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Thm.(D-Meyer) There is an exact sequence

$$\mathsf{Ext}_{\mathsf{C}(X,\Lambda)}\big(\underline{\mathsf{K}}(A),\underline{\mathsf{K}}_*(SB)\big) \rightarrowtail \mathsf{KK}_X(A,B) \twoheadrightarrow \mathsf{Hom}_{\mathsf{C}(X,\Lambda)}\big(\underline{\mathsf{K}}(A),\underline{\mathsf{K}}(B)\big).$$